A Brief Overview of Measure Theory

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Definition

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- A measure is an extended real valued non-negative, and countably additive set function (μ) defined on a ring (R) and such that $\mu(\emptyset) = 0$.
- Taking the ring to be points on the number line as depicted in Figure 1

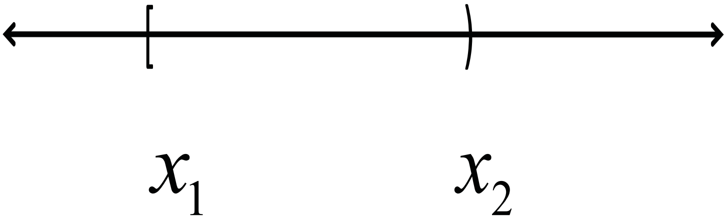


Figure: Borel Field on the Real Number Line

- Assume the set mapping function from these sets of xs to the real number line

$$\begin{array}{c} \mu\left(\{\ x\}\ \right)=\frac{1}{2}\left(x_j-x_i\right)^2 \\ \mu : X \to R_1 \end{array} \tag{1}$$

- Taking a couple of examples
  - Looking at unit intervals

$$\begin{array}{c} E_1=[x_1,x_2)=[1,2)\Rightarrow\mu\left(E_1\right)=\frac{1}{2} \\ E_2=[2,3)\Rightarrow\mu\left(E_2\right)=\frac{1}{2} \\ E_3=[3,4)\Rightarrow\mu\left(E_3\right)=\frac{1}{2} \\ E_4=[4,t)\Rightarrow\mu\left(E_4\right)=\frac{1}{2} \end{array} \tag{2}$$

- Reconstructing the set function on two unit intervals

$$\begin{array}{c} E_1^* = [1,3) \Rightarrow \mu\left(E_1^*\right) = 2 \\ E_2^* = [3,4) \Rightarrow \mu\left(E_2^*\right) = 2 \end{array} \tag{3}$$

- Reconstructing the set function on a four unit interval

$$E_1^{**} = [1,5) \Rightarrow \mu\left(E_1^{**}\right) = 8 \tag{4}$$

- Can this set function be a measure?
  - The function is a positive real valued function.
  - However, it is not countably additive

$$\begin{array}{ccc} \mu\left(E_1^{**}\right) & > \mu\left(E_1^*\right) + \mu\left(E_2^*\right) & > \mu\left(E_1\right) + \mu\left(E_2\right) + \mu\left(E_3\right) + \mu\left(E_4\right) \\ 8 & > 2 + 2 & > 1/2 + 1/2 + 1/2 + 1/3 \end{array} \tag{5}$$

- Trivial case of measure

$$\mu\left(\{\ x\}\ \right)=\left|x_j-x_i\right| \tag{6}$$

or the standard distance formula

$$\mu^*\left(\{\ x\}\ \right)=\left[\left(x_j-x_i\right)\right]^{1/2} \tag{7}$$

- The function values for the set mapping functions

$$\begin{aligned} E_1 &= [1,2) \Rightarrow \mu^*\left(E_1\right)=1 \\ E_2 &= [2,3) \Rightarrow \mu^*\left(E_2\right)=1 \\ E_3 &= [3,4) \Rightarrow \mu^*\left(E_3\right)=1 \\ E_4 &= [4,5) \Rightarrow \mu^*\left(E_4\right)=1 \end{aligned} \tag{8}$$

$$\begin{aligned} E_1^* &= [1,3) \Rightarrow \mu^*\left(E_1^*\right)=2 \\ E_2^* &= [3,5) \Rightarrow \mu^*\left(E_2^*\right)=2 \end{aligned} \tag{9}$$

$$E_1^{**} = [1,5) \Rightarrow \mu^*\left(E_1^{**}\right) = 4 \tag{10}$$

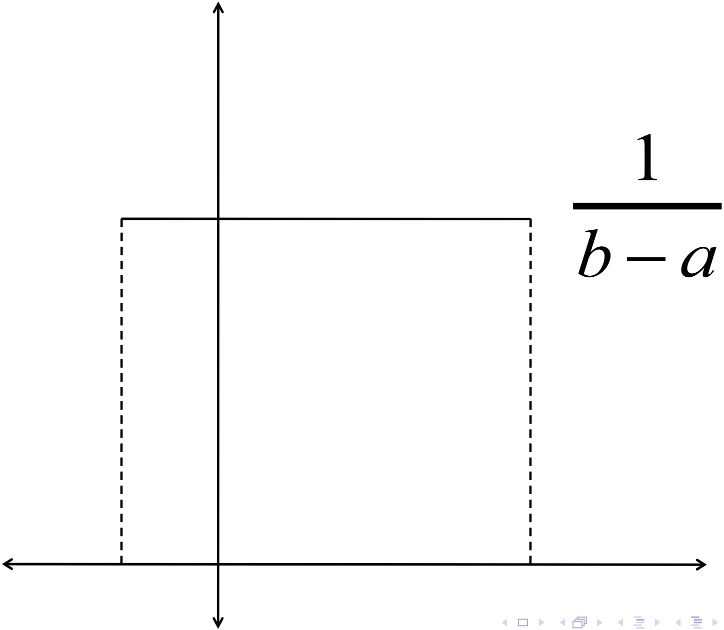
$$\begin{array}{ccc} \mu^*\left(E_1^{**}\right) & = \mu^*\left(E_1^{*}\right) + \mu^*\left(E_2^{*}\right) & = \mu^*\left(E_1\right) + \mu^*\left(E_2\right) + \mu^*\left(E_3\right) + \mu^*\left(E_4 \right. \\ 4 & = 2 + 2 & = 1 + 1 + 1 + 1 \end{array} \tag{11}$$

Correspondence with Probability Function

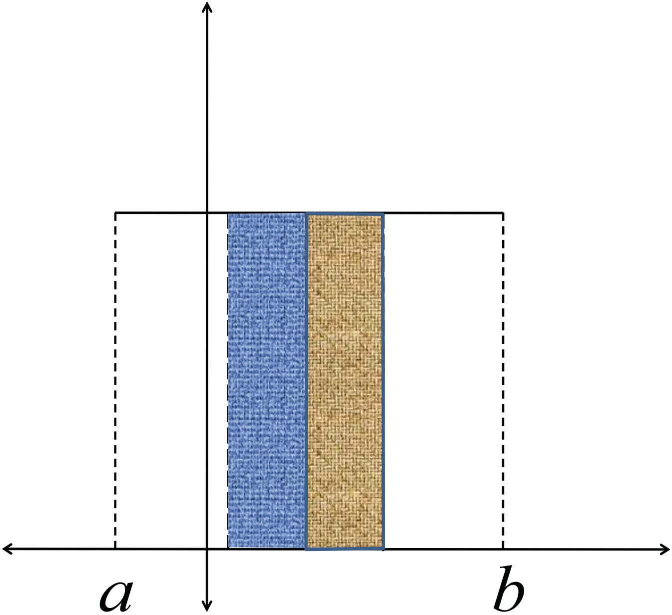
- Let $P\left[\{\ x\}\ \right]$ be a measure function defined on a portion of the real number line (i.e., $x \in [a, b]$)
- The measure function on this interval is then defined as

$$
\begin{aligned}
P\left[[x_i, x_j)\right] &= \int_{\max(x_i,a)}^{\min\left(x_j,b\right)} \frac{1}{b-1} dx \\
&= \frac{1}{b-a}\left(\min\left(x_j, b\right) - \max\left(x_i, a\right)\right)
\end{aligned} \tag{12}
$$

- This is a real-valued set function, whose result is positive.


$\frac{1}{b-a}$

- Graphically, the countably additive characteristics can be demonstrated in Figure.


a
b